Last time : SVD of $B \in \mathbb{R}^{m\times n}$

$$B = U \Sigma V^{-1}$$

$$= \underset{m\times m}{U} \; \underset{m\times n}{\Sigma} \; V^T_{n\times n}$$

where U and V are orthogonal

$$\Sigma = \begin{pmatrix} \sigma_1 & & & 0 & \\ & \ddots & & & \\ & & \sigma_r & & 0 \\ & 0 & & 0 & \\ & & & & 0 \end{pmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$

↳ singular values of B.

Geometric interpretation : $\Sigma e_i = \begin{cases} \sigma_i e_i & \text{if } i \leq r \\ 0 & \text{if } i > r \end{cases}$.

diagonal

Σ

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, bend left] & \mathbb{R}^m \arrow[d, "U"] \\
\mathbb{R}^n \arrow[u, "V^{-1}"] \arrow[r, bend left, "B"] & \mathbb{R}^m
\end{tikzcd}
$$

rotation   rotation

Application of diagonalization : solving systems of linear ODE (ordinary differential equations)

derivative w.r.t t

$$
\begin{cases}
x_1'(t) = a_{11}x_1(t) + \dots + a_{1n}x_n(t) \\
\vdots \qquad\qquad \vdots \\
x_n'(t) = a_{n1}x_1(t) + \dots + a_{nn}x_n(t)
\end{cases}
$$

known coefficients

Solve for $x_1(t), \dots, x_n(t) : \mathbb{R} \to \mathbb{R}$ (t is "time")

subject to an initial condition $x_1(0) = c_1, \dots, x_n(0) = c_n$

Warm-up: recursion (Fibonacci numbers)

$$F_0, F_1, F_2, F_3, F_4, F_5, \dots\dots, F_n, \dots\dots$$

0 1 1 2 3 5

- recursion: $F_n = F_{n-1} + F_{n-2}$, $\forall n \geq 2$

order 2, because F_n depends on previous two : F_{n-1}, F_{n-2}

- initial condition : $F_0 = 0$, $F_1 = 1$

Reduce to a recursion of order 1:

$$G_n = F_{n-1}$$

$$\mathbb{R}^2 \ni \underbrace{\begin{pmatrix} F_n \\ G_n \end{pmatrix}}_{v_n} = \begin{pmatrix} F_{n-1} + F_{n-2} \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n-1} + G_{n-1} \\ F_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} F_{n-1} \\ G_{n-1} \end{pmatrix}}_{v_{n-1}}$$

recursion of order 1

Matrix form: $v_n = A v_{n-1}$

initial condition : $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ Solve for v_n :

$$v_n = A v_{n-1} = A A v_{n-2} = A A A v_{n-3} = \ldots\ldots = \underbrace{A A \ldots A}_{n-1 \text{ times}} v_1 = A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Upshot: finding F_n boils down to calculating A^{n-1}

To calculate A^{n-1} diagonalize $A = PDP^{-1}$

To calculate A , diagonalize A

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

$$A^3 = PD^2P^{-1}PDP^{-1} = PD^3P^{-1}$$

$$\vdots$$

$$A^{n-1} = \dots\dots = PD^{n-1}P^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \rightsquigarrow D^{n-1} = \begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}$$

So we have to diagonalize $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$\chi_A(t) = \det\begin{pmatrix} t-1 & -1 \\ -1 & t \end{pmatrix} = t^2 - t - 1$$

$$\rightsquigarrow \lambda_1 = \frac{1+\sqrt{5}}{2}$$ (golden ratio)

$$\rightsquigarrow \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$v_1 \in \operatorname{Ker}(A - \lambda_1 \cdot I_2) = \operatorname{Ker}\begin{pmatrix} +\frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix}$$

subtract $-\frac{1}{2} - \frac{\sqrt{5}}{2}$ row_1 from row_2

$$= \mathrm{Ker}\begin{pmatrix} \frac{1}{2}-\frac{\sqrt{5}}{2} & 1 \\ \underset{0}{\cancel{1-\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}} & 0 \end{pmatrix}$$

choose $v_1 = \begin{pmatrix} 1 \\ \frac{\sqrt{5}}{2}-\frac{1}{2} \end{pmatrix}$

similarly $v_2 = \begin{pmatrix} 1 \\ -\frac{\sqrt{5}}{2}-\frac{1}{2} \end{pmatrix}$

$$A = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}^{-1}$$

$$\left(\text{recall: } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right)$$

$$P^{-1} = \frac{1}{-\sqrt{5}}\begin{pmatrix} \frac{-\sqrt{5}-1}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{2}-\frac{1}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$A^{n-1} = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}\begin{pmatrix} \frac{1}{2}+\frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{2}-\frac{1}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$v_n = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt5-1}{2} & \frac{-\sqrt5-1}{2} \end{pmatrix} \begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix} \underbrace{\begin{pmatrix} \frac12 + \frac{1}{2\sqrt5} & \frac{1}{\sqrt5} \\ \frac12 - \frac{1}{2\sqrt5} & -\frac{1}{\sqrt5} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\|$$

$$\begin{pmatrix} F_n \\ G_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ \frac{\sqrt5-1}{2} & \frac{-\sqrt5-1}{2} \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix} \begin{pmatrix} \frac12 + \frac{1}{2\sqrt5} \\ \frac12 - \frac{1}{2\sqrt5} \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 1 \\ \frac{\sqrt5-1}{2} & \frac{-\sqrt5-1}{2} \end{pmatrix} \begin{pmatrix} \lambda_1^{n-1}\left(\frac12 + \frac{1}{2\sqrt5}\right) \\ \lambda_2^{n-1}\left(\frac12 - \frac{1}{2\sqrt5}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1^{n-1}\left(\frac12 + \frac{1}{2\sqrt5}\right) + \lambda_2^{n-1}\left(\frac12 - \frac{1}{2\sqrt5}\right) \\ \cdots\cdots \end{pmatrix} = \begin{pmatrix} F_n \\ G_n \end{pmatrix}$$

Upshot: $$F_n = \lambda_1^{n-1}\underbrace{\left(\frac12 + \frac{1}{2\sqrt5}\right)}_{\frac{1+\sqrt5}{2\sqrt5} = \frac{\lambda_1}{\sqrt5}} + \lambda_2^{n-1}\underbrace{\left(\frac12 - \frac{1}{2\sqrt5}\right)}_{\frac{\sqrt5-1}{2\sqrt5} = -\frac{\lambda_2}{\sqrt5}}$$

$$\boxed{F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt5}}, \text{ where } \lambda_1 = \frac{1+\sqrt5}{2}, \quad \lambda_2 = \frac{1-\sqrt5}{2}$$

Check: $F_0 = \frac{1-1}{\ } = 0$ ✓

$\sqrt{5}$

$$F_1 = \frac{\lambda_1 - \lambda_2}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \checkmark$$

$\vdots$

$$F_3 = \frac{\lambda_1^3 - \lambda_2^3}{\sqrt{5}} = \frac{(1+\sqrt{5})^3 - (1-\sqrt{5})^3}{8\sqrt{5}} = \frac{\begin{matrix} 1 \\ +3\sqrt{5} \\ +3\cdot 5 \\ +5\sqrt{5} \end{matrix} - \begin{matrix} 1 \\ -3\sqrt{5} \\ 3\cdot 5 \\ -5\sqrt{5} \end{matrix}}{8\sqrt{5}} = \frac{6\sqrt{5}+10\sqrt{5}}{8\sqrt{5}} = 2$$

$\vdots$

Now: solve systems of ODE

order 1

$$\begin{cases} x'(t) = \underset{}{\overset{\mathbb{R}\ni}{a}}\, x(t) & \text{for some } a \in \mathbb{R} \\ x(0) = \underset{\in\mathbb{R}}{c} \end{cases}$$

unknown function

Postulate: $\boxed{x(t) = e^{at} c}$

$x'(t) = a e^{at} c = a x(t)$

$x(0) = e^{0t} c = c$

· order 3 ODE (because 3 derivatives appear):

$$\begin{cases} x'''(t) = 4x''(t) - 3x'(t) + 9x(t) \\ x(0) = 17, \quad x'(0) = -5, \quad x''(0) = 19 \end{cases}$$

convert to order 1: $Y(t) = x'(t)$, solve for $v(t) = \begin{pmatrix} x(t) \\ Y(t) \\ \ldots \end{pmatrix}$

$$z(t) = x''(t)$$

$$v'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ z(t) \\ x'''(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ z(t) \\ 4z(t) - 3y(t) + 9x(t) \end{pmatrix}$$

$$v'(t) = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 9 & -3 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = A\,v(t), \quad v(0) = \begin{pmatrix} 17 \\ -5 \\ 19 \end{pmatrix}$$

Problem of finding $x(t)$ boils down to solving

$$v'(t) = \underbrace{A}_{\in \mathbb{R}^{3\times 3}}\, \underbrace{v(t)}_{\in \mathbb{R}^3,\ \text{depending on } t}, \quad v(0) = c \in \mathbb{R}^3$$

Problem: solve $v'(t) = \underbrace{A}_{\in \mathbb{R}^{n\times n}} v(t)$, $v(0) = C \in \mathbb{R}^n$

A is known, c is known, solve for $v(t)$.

Postulate: $\boxed{v(t) = e^{At} C}$ is the solution

What on Earth is e^A, where A is a square matrix?

Well, what is e^a, where $a \in \mathbb{R}$?

$\Vert$ Taylor series

$$1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \cdots\cdots$$

DEF: for any $n \times n$ matrix $A \in \mathbb{R}^{n \times n}$

$$e^A = \exp(A) := I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots \in \mathbb{R}^{n \times n}$$

= a convergent series of $n \times n$ matrices

Properties • $e^A e^B = e^{A+B} = e^B e^A$, $\underline{\text{if } AB = BA}$

0 matrix

• $e^0 = I_n$

• e^A is invertible and its inverse is e^{-A}

$(e^A e^{-A} = e^{A-A} = e^0 = I_n)$

• $D = \begin{pmatrix} d_1 & & \\ & d_2 & O \end{pmatrix}$ and $e^D = \begin{pmatrix} e^{d_1} & & \\ & e^{d_2} & O \end{pmatrix}$

$\cdot\ D = \begin{pmatrix} \ddots & \\ 0 & d_n \end{pmatrix}$ ... $\begin{pmatrix} & \ddots \\ 0 & e^{d_n} \end{pmatrix}$

$A \sim B$

$\cdot$ if $A = PBP^{-1}$ $\forall A, B \in \mathbb{R}^{n\times n}$, P invertible

then $e^A = P e^B P^{-1}$

Proof:

$$\begin{aligned}
I_n &= P I_n P^{-1} \\
A &= P B P^{-1} \\
\frac{A^2}{2!} &= P \frac{B^2}{2!} P^{-1} \\
\frac{A^3}{3!} &= P \frac{B^3}{3!} P^{-1} \\
&\ \ \vdots \\
\hline
e^A &= P\, e^B\, P^{-1} \quad (+)
\end{aligned}$$

$\square$

THM 27.1 $\begin{cases} v'(t) = A\, v(t) \\ v(0) = C \end{cases}$ has solution $\boxed{v(t) = e^{At} C}$

known

... $\mathbb{R}^{n\times n}$ ... e^{At} ?

Practically, given $A \in \mathbb{R}^{n \times n}$, how to calculate e^{At}?

Answer: diagonalize $A = PDP^{-1}$

$$e^{At} = Pe^{Dt}P^{-1}$$

$$\Rightarrow v(t) = \underset{n\times n}{P}\,\underset{n\times n}{e^{Dt}}\,\underset{n\times n}{P^{-1}}\,\underset{n\times 1}{C} \in \mathbb{R}^n \text{ depending on } t$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

so all terms in the formula above for $v(t)$ can be calculated

Ex: order 2 ODE : $\begin{cases} x''(t) = x'(t) + x(t) \\ x(0) = 0,\ x'(0) = 1 \end{cases}$

Step 1: convert to system of order 1 ODE

↳ only first derivatives

Let $Y(t) = x'(t)$

$$v(t) = \begin{pmatrix} Y(t) \\ x(t) \end{pmatrix}$$

$$v'(t) = \begin{pmatrix} y'(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} x''(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x'(t) + x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) + y(t) \\ y(t) \end{pmatrix}$$

$$v'(t) = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} y(t) \\ x(t) \end{pmatrix} = A\, v(t) \quad \Big| \quad v(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Step 2: formula (Thm 27.1) $v(t) = e^{At} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Step 3: diagonalize A

$$A = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}^{-1}$$

$$\Downarrow$$

$$e^{At} = Pe^{Dt}P^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{2} - \frac{1}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

Step 4: Calculate answer:

$$v(t) = \begin{pmatrix} 1 & 1 \\ \cdots \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ \cdots \end{pmatrix} \begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{5}} & \cdots \end{pmatrix}$$

$$v(t) = e^{At}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{5}} \\ \frac{1}{2} - \frac{1}{2\sqrt{5}} \end{pmatrix}$$

$$\|$$

$$\begin{pmatrix} Y(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t}\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right) \\ e^{\lambda_2 t}\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right) \end{pmatrix}$$

$$\begin{pmatrix} Y(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} \cdots\cdots \\ e^{\lambda_1 t}\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)\left(\frac{\sqrt{5}-1}{2}\right) + e^{\lambda_2 t}\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)\left(\frac{-\sqrt{5}-1}{2}\right) \end{pmatrix}$$

$$\Downarrow$$

$$x(t) = e^{\lambda_1 t} \cdot \frac{\lambda_1}{\sqrt{5}}(-\lambda_2) + e^{\lambda_2 t}\frac{(-\lambda_2)}{\sqrt{5}}(-\lambda_1)$$

$$= \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\sqrt{5}}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\Downarrow$$

$$\lambda_1 \lambda_2 = -1$$

Check above solution:

- initial condition $x(0) = \frac{1-1}{\sqrt{5}} = 0$, $x'(0) = \frac{\lambda_1 - \lambda_2}{\sqrt{5}} = 1$
- ODE $x''(t) = x'(t) + x(t)$

$$\left.\begin{aligned} x(t) &= \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\sqrt{5}} \\ x'(t) &= \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\sqrt{5}} \end{aligned}\right\} \quad x(t) + x'(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t} + \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\sqrt{5}}$$

$$\|$$

$$x''(t) = \lambda_1^2 e^{\lambda_1 t} - \lambda_2^2 e^{\lambda_2 t} \quad\quad (\lambda_1 + 1)e^{\lambda_1 t} - (\lambda_2 + 1)e^{\lambda_2 t}$$

$$x''(t) = \frac{\lambda_1 e^{\cdots} - \lambda_2 e^{\cdots}}{\sqrt{5}} = \frac{(\lambda_1 + 1) e^{\cdots} - (\cdots)}{\sqrt{5}}$$

because
$\lambda_1^2 = \lambda_1 + 1$
$\lambda_2^2 = \lambda_2 + 1$ (this is related to the fact that λ_1, λ_2 are roots of $\chi_A(t)$)

Why does the main theorem hold?

$$\begin{cases} v'(t) = A\, v(t) \\ v(0) = C \end{cases} \rightsquigarrow \boxed{v(t) = e^{At} C}$$

Because the formula in the blue box satisfies the

- initial condition $v(0) = e^{A0} C = I_n C = C$
- ODE $v'(t) = A\, v(t)$

$$\Uparrow$$

$$(e^{At} c)' = A e^{At} c$$

$$\Uparrow$$

$$(e^{At})' = A e^{At} \quad \circledast$$

Matrix derivatives

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}' = \begin{pmatrix} a_{11}'(t) & a_{12}'(t) \\ a_{21}'(t) & a_{22}'(t) \end{pmatrix}$$

Proof of $\circledast$: $e^{At} = I_n + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \dots$

$$(e^{At})' = 0 + A + A^2 t + \frac{A^3 t^2}{2!} + \frac{A^4 t^3}{3!} + \dots$$

$$= A\left(I_n + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots\right)$$

$$= A e^{At} \qquad \square$$

Ex: solve system of three order 1 ODEs

$$\begin{cases} x'(t) = x(t) + y(t) \\ y'(t) = x(t) + y(t) \\ z'(t) = 3z(t) \end{cases} \qquad \begin{aligned} x(0) &= 2 \\ y(0) &= 4 \\ z(0) &= 0 \end{aligned}$$

$v(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \in \mathbb{R}^3$ which depends on t

$$v'(t) = \begin{pmatrix} x(t) + y(t) \\ x(t) + y(t) \\ 3z(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = A v(t)$$

$$v(0) = \underbrace{\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}_{C}$$

Thm 27.1 $\Rightarrow$ $\boxed{v(t) = e^{At} C}$

Diagonalize A: eigenvalues are $3, 2, 0$

eigenvectors are $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$A = PDP^{-1}$, where $D = \begin{pmatrix} 3 & & 0 \\ & 2 & \\ 0 & & 0 \end{pmatrix}$

$P = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$

$\Rightarrow v(t) = P e^{Dt} P^{-1} C$

$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x(t) \\ \cdots \end{pmatrix} = \begin{pmatrix} 3e^{2t} - 1 \\ \cdots e^{2t} \end{pmatrix}$

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 3\,e^{\dots} + 1 \\ 0 \end{pmatrix}$$

$\square$

Ex: Solve $x''(t) = -x(t)$, $x(0) = 1$, $x'(0) = 0$

Let $y(t) = x'(t)$ and solve for $v(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$v'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -x(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} v(t)$$

$$v(0) = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{C}$$

Thm 27.1 $\Longrightarrow$ $\boxed{v(t) = e^{At}\,C}$

A has conjugate eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$

eigenvectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\Downarrow$

$\dots P^{-1} \dots \begin{pmatrix} i & 0 \end{pmatrix}$

$A = PDP$, where $D = \begin{pmatrix} & \\ 0 & -i \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$e^{At} = P e^{\Delta t} P^{-1}$$

$$P^{-1} = \frac{1}{-2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$e^{At} = P \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} P^{-1}$$

$$v(t) = e^{At} c = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{it} \\ e^{-it} \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ \cdots \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{it} + e^{-it} \\ \cdots\cdots \end{pmatrix}$$

$\Rightarrow$ Solution to $x''(t) = -x(t)$, $x(0) = 1$, $x'(0) = 1$ is $x(t) = \dfrac{e^{it} + e^{-it}}{2}$

But $x(t) = \cos(t)$ is also a solution because it works!

But $x(t) = \cos(t)$ is also a solution, because of

$$x'(t) = -\sin(t)$$

$$x''(t) = -\cos(t)$$

$\cos(0)=1$
$\sin(0)=0$

Upshot:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

derivative and multiply by $-i$

$$i \cdot \sin(t) = \frac{e^{it} - e^{-it}}{2}$$

(+)

$$\cos(t) + i\sin(t) = e^{it}$$

Euler's formula

set $t=\pi$: $-1 = e^{i\pi}$

definition

(more generally, $e^{a+bi} = e^a e^{bi} = e^a(\cos(b) + i\sin(b))$)

THM 28.1 (EULER): $e^{it} = \cos(t) + i\sin(t)$

Proof by Taylor series:

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \dots$$

1
i
$i^2 = -1$
$i^3 = -i$
$i^4 = -i^2 = 1$
$i^5 = i$

$$e^{it} = 1 \quad - \frac{t^2}{2!} \quad + \frac{t^4}{4!} \quad - \dots \quad = \cos(t)$$

$$+$$

$$+it \quad - i\frac{t^3}{3!} \quad + i\frac{t^5}{5!} \quad + \dots \quad = i \sin(t)$$

$i^6 = -1$

$i^7 = -i$

$i^8 = 1$

$\vdots$

In general, the solution to systems of linear ODEs (of any order) may be written in terms of

- real exponentials (e^{at})
- sines and cosines ($\cos(bt)$, $\sin(bt)$)
- various constants

complex exponentials $e^{(a+bi)t}$

Here's how to set up such a complicated system using matrices:

$$\begin{cases} x'''(t) = 5y'(t) + x''(t) - 3y(t) + 6x(t) \\ y''(t) = 2x''(t) - 4x'(t) + 17\,y(t) \end{cases}$$

Introduce new symbols for the various derivatives:

$x_0(t) = x(t)$ $\qquad$ $y_0(t) = y(t)$

$x_1(t) = x'(t)$ $\qquad$ $y_1(t) = y'(t)$

$x_2(t) = x''(t)$

and solve for $v(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ y_0(t) \\ y_1(t) \end{pmatrix} \in \mathbb{R}^5$

This big guy satisfies the order 1 ODE

$$v'(t) = \begin{pmatrix} x_0'(t) \\ x_1'(t) \\ x_2'(t) \\ y_0'(t) \\ y_1'(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ 6x_0(t) + x_2(t) - 3y_0(t) + 5y_1(t) \\ y_1(t) \\ -4x_1(t) + 2x_2(t) + 17y_0(t) \end{pmatrix}$$

$$\Downarrow$$

$$v'(t) = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -4 & 2 & 17 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ y_0(t) \\ y_1(t) \end{pmatrix} = Av(t)$$

Thm 27.1 $\Downarrow$

$v(t) = e^{At} \cdot$ initial condition

and you may calculate e^{At} by diagonalizing A.

Please evaluate our course on Moodle; it means a lot to us!